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Nonlinear waves in compacting media

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An investigation of the mathematical model of a compacting medium proposed by McKenzie (1984) for the purpose of understanding the migration and segregation of melts in the Earth is presented. The numerical observation that the governing equations admit solutions in the form of nonlinear one-dimensional waves of permanent shape is confirmed analytically. The properties of these solitary waves are presented, namely phase speed as a function of melt content, nonlinear interaction and conservation quantities. The information at hand suggests that these waves are not solitons.

1. Introduction

This paper arises from our attempt to understand the geophysical problem associated with melt segregation. The problem can be stated thus: large volumes of molten rock are extruded onto the surface or exist as magma chambers, yet laboratory studies of Earth materials at pressures and temperatures comparable with those in the Earth's interior show that at best only a small degree of partial melting can occur (typically less than 10 %). The geophysicists are therefore forced to conclude that the molten rocks that have erupted onto the surface or in magma chambers must have segregated or migrated over large distances.

How does this segregation/migration process take place? In an attempt to answer this question, mathematical models of the migration process have been proposed (Sleep 1974; Walker, Stolper & Hays 1978). The one that we shall use is that of McKenzie (1984) together with further elaborations by Richter & McKenzie (1984). In this model, both the solid matrix and the fluid melt are treated as two immiscible fluids of constant, but different, densities. Since these two fluids interpenetrate each other and their contact surface is extremely convoluted, the approach of two-phase fluid mechanics is adopted. In particular, at any given point in the medium, a field $\phi(x, t)$ is defined which represents the volume fraction of melt. This field is analogous to the traditional porosity in the theory of flows through porous media. Another characteristic of flows through porous media is the permeability K, which represents a measure of the ability of the fluid to traverse the porous region. An essential feature of the McKenzie model is that the permeability K is related to the voidage ϕ by means of a power law

$$K = K_0 \phi^n.$$

The flow of melt in a given region is more or less impeded by the solid matrix, depending upon the value of the voidage in that region.

At this early stage of its development, the model does not account for changes of phase. In other words, it is assumed that the rocks have already partially melted and only the subsequent migration of the melt under the action of gravity is considered.

This paper is concerned with the one-dimensional version of the model studied by

Richter & McKenzie (1984). In that paper, as well as in Scott & Stevenson (1984), it was noticed that arbitrary initial distributions of voidage broke up into a series of rank-ordered, finite-amplitude waves of permanent shape. We shall investigate, both analytically and numerically, the properties of these waves in some detail. In particular, we shall exhibit their shapes and study their speed of propagation as a function of amplitude and background properties. Finally, we shall examine the question of whether or not these solitary waves are solitons. The designation of solitons is reserved for solitary waves that have the peculiar property of going through a nonlinear interaction unscathed. It is widely believed that this property is associated with the fact that solitons satisfy an infinite number of conservation laws. Therefore, we have performed some numerical experiments involving two interacting waves and have initiated a systematic search for conservation laws. The preliminary conclusions are that these solitary waves are *not* solitons.

2. The mathematical model

A microscopic picture of the matrix/melt system is given schematically in figure 1. If the rock is sliced and viewed in plan, the melt appears confined to those places where three grain boundaries meet. In three dimensions, however, the melt can form an interconnected network around the grains as shown in part (*b*) of the figure. The degree of melting required for the melt phase to be completely interconnected depends on the surface energies, and hence on the composition of the original rock. In the case of basaltic systems, which are the most common volcanic source, it is believed that the melt remains interconnected down to vanishingly small degrees of melt (McKenzie 1984).

In order to simplify the analysis, we adopt a macroscopic two-phase approach for the description of the matrix/melt system. We refer the reader to McKenzie (1984) and Richter & McKenzie (1984) for a full discussion of the assumptions and approximations underlying the model that we shall use. Here, we ignore the difficulties concerning two-phase fluid flows and reduce the derivation of the governing equations to a formal correspondence between microscopic and macroscopic fields. We shall denote macroscopic fields by superscripts (1) for the melt and (2) for the solid matrix; microscopic fields will have subscripts m and s. At the heart of the correspondence is an averaging procedure over a volume of dimension both much smaller than any macroscopic characteristic length and much larger than the microscopic lengthscale. Keeping this averaging control volume in mind, we write the densities as

$$\left.\begin{aligned} \rho^{(1)} &= \phi\rho_{\mathrm{m}}, \\ \rho^{(2)} &= (1-\phi)\rho_{\mathrm{s}}, \end{aligned}\right\} \tag{2.1}$$

where ϕ is the volumeric fraction of melt. As is typical with macroscopic descriptions, both $\rho^{(1)}$ and $\rho^{(2)}$ can be evaluated at the same point $\boldsymbol{x}$. Momentum considerations enable us to write

$$\rho^{(1)}\boldsymbol{v}^{(1)} = \phi\rho_{\mathrm{m}}\boldsymbol{v}_{\mathrm{m}},$$

$$\rho^{(2)}\boldsymbol{V}^{(2)} = (1-\phi)\rho_{\mathrm{s}}\boldsymbol{V}_{\mathrm{s}},$$

and hence

$$\left.\begin{aligned} \boldsymbol{v}^{(1)} &= \boldsymbol{v}_{\mathrm{m}}, \\ \boldsymbol{V}^{(2)} &= \boldsymbol{V}_{\mathrm{s}}. \end{aligned}\right\} \tag{2.2}$$

In the following, we shall dispense with subscripts and superscripts for the velocity fields: lower and upper case $\boldsymbol{v}$ and $\boldsymbol{V}$ will denote both the microscopic and macroscopic velocity field of the melt and solid matrix respectively.

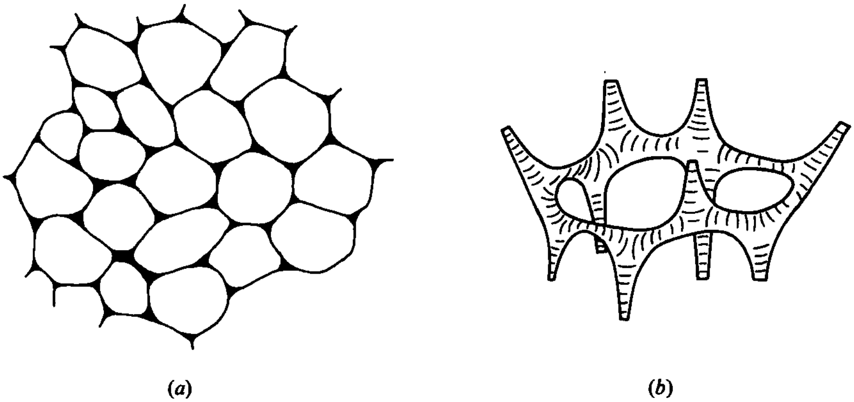

FIGURE 1. Schematic diagrams of melt distribution. (a) Solid grains with interstitial melt along boundaries where three grains meet. (b) The fluid phase as a three-dimensional network of interconnected melt tubes.

Finally, from force considerations, we have

$$\left.\begin{aligned}\sigma_{ij}^{(1)} &= \phi\sigma_{\mathrm{m}ij},\\ \sigma_{ij}^{(2)} &= (1-\phi)\,\sigma_{\mathrm{s}ij},\end{aligned}\right\} \tag{2.3}$$

where σ_{ij} is the stress tensor. The balance of forces for each phase is written as follows:

$$\frac{\partial\sigma_{kl}^{(i)}}{\partial x_l}+I^{(i)}+\rho^{(i)}F_k^{(i)} = 0.$$

In the above equations, $F^{(i)}$ stands for the body force per unit mass in the ith phase, whereas $I^{(i)}$ is the extra force arising from the two-phase nature of the flow (Drew & Segel 1971). Note that the inertial forces have been neglected. The only body forces in the problem are gravitational:

$$\left.\begin{aligned}\rho^{(1)}F_k^{(1)} &= -\phi\rho_{\mathrm{m}}\,g\delta_{k3},\\ \rho^{(2)}F_k^{(2)} &= -(1-\phi)\,\rho_{\mathrm{s}}\,g\delta_{k3},\end{aligned}\right\} \tag{2.4}$$

where g is the gravitational acceleration and δ_{ij} is the Kronecker delta.

The next step consists in specifying constitutive relations. Those used by McKenzie are:

$$I_k^{(1)} = -I_k^{(2)} = -p_{\mathrm{m}}\frac{\partial}{\partial x_k}(1-\phi)-\frac{\mu\phi^2}{K}(v_k-V_k), \tag{2.5}$$

$$\sigma_{\mathrm{m}ij} = -p_{\mathrm{m}}\,\delta_{ij}, \tag{2.6}$$

$$\sigma_{\mathrm{s}ij} = -p_{\mathrm{m}}\,\delta_{ij}+\xi^*\frac{\partial V_k}{\partial x_k}\delta_{ij}+\eta^*\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}-\frac{2}{3}\frac{\partial V_k}{\partial x_k}\delta_{ij}\right). \tag{2.7}$$

Thus the interaction is a function of the pressure p_{m} in the melt, the voidage ϕ, the relative velocity between the two phases and the permeability K. The stresses in the melt are isotropic and proportional to the pressure. However, this does not mean that the melt is an inviscid fluid. In fact its viscosity μ enters explicitly in (2.5). The solid matrix is treated as a viscous, compressible non-Stokesian fluid with bulk viscosity ξ^* and shear viscosity η^*. The working versions of the momentum equations become

$$\phi(\boldsymbol{v}-\boldsymbol{V})+\mu^{-1}K\,\boldsymbol{\nabla}P = 0, \tag{2.8}$$

$$\eta\,\nabla^2\boldsymbol{V}+(\xi+\tfrac{1}{3}\eta)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{V})-\boldsymbol{\nabla}P-(\rho_{\mathrm{s}}-\rho_{\mathrm{m}})\,g(1-\phi)\,\boldsymbol{k} = 0. \tag{2.9}$$

In arriving at these equations, it was assumed that

$$\xi = (1-\phi)\,\xi^*,$$
$$\eta = (1-\phi)\,\eta^*$$

are constants and that
$$P = p_{\mathrm{m}} + \rho_{\mathrm{m}} g z.$$

The formulation is completed by the conservation-of-mass equations, viz

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \phi v = 0, \tag{2.10}$$

$$-\frac{\partial \phi}{\partial t} + \nabla \cdot (1-\phi)\, V = 0, \tag{2.11}$$

and the power law
$$K = K_0 \phi^n. \tag{2.12}$$

Values of n of around 3 are typical for the geophysical situations of interest.

We shall non-dimensionalize the various fields as follows:

$$\left.\begin{aligned} \phi &= \phi_0 \phi', \\ K &= K_0 K', \\ (W, w) &= \mu^{-1} K_0 (1-\phi_0)\, \Delta\rho\, g (W', w'), \\ z &= \{\mu^{-1} K_0 (\xi + \tfrac{4}{3}\eta)\}^{\frac{1}{2}} z', \\ t &= \frac{1}{\phi_0} \left\{ \frac{\mu(\xi + \frac{4}{3}\eta)}{K_0 (1-\phi_0)\, \Delta\rho\, g} \right\}^{\frac{1}{2}} t', \end{aligned}\right\} \tag{2.13}$$

where $\Delta\rho = \rho_{\mathrm{s}} - \rho_{\mathrm{m}}$ and W, w denote respectively the vertical velocity in the solid phase and in the melt. Dropping the primes and eliminating P and w, the non-dimensional equations reduce to

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z} (1 - \phi_0 \phi)\, W, \tag{2.14}$$

$$\frac{\partial^2 W}{\partial z^2} - \frac{W}{\phi^n} - \frac{1-\phi_0 \phi}{1-\phi_0} = 0. \tag{2.15}$$

The above equations are identical to those given in Richter & McKenzie (1984) except for the non-dimensionalization of time. With the present non-dimensionalization there exists an interior solution $\phi = 1$, $W = -1$ which represents a uniform, compacting, free subsidence of solid matrix relative to the melt. In our analysis of solitary waves, we shall assume this solution as the background state on which the disturbances are superimposed.

3. Solitary waves

Figure 2 shows an initial distribution of voidage $\phi(z, 0)$ as well as its evolution at various later times. These profiles have been obtained by numerically integrating (2.14)–(2.15) for $n = 3$. The numerical methods are discussed in Appendix A. As we can see, the original disturbance breaks up into a series of spikes which are ordered in rank and which propagate without apparent subsequent change in shape. This numerical experiment is typical of several we have run and strongly suggests the existence of solitary waves of permanent form.

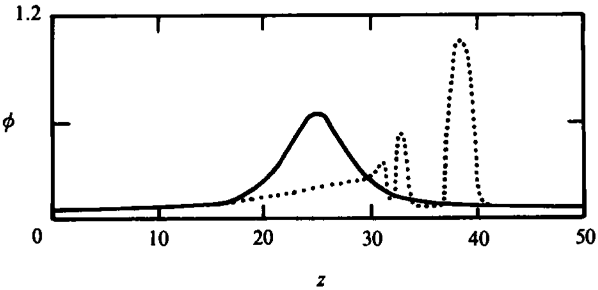

FIGURE 2. Fluid fraction ϕ as a function of depth. The solid curve is the initial voidage distribution which evolves into a series of discrete pulses (dashed curve) suggesting the existence of solitary waves.

We look for such waves by searching for solutions of the form

$$\left.\begin{aligned} \phi(z,t) &= f(z-ct), \\ W(z,t) &= g(z-ct). \end{aligned}\right\} \tag{3.1}$$

The governing equations become

$$-cf' = [(1-\phi_0 f)\,g]', \tag{3.2}$$

$$g'' - \frac{g}{f^n} - \frac{1-\phi_0 f}{1-\phi_0} = 0, \tag{3.3}$$

where primes denote differentiation with respect to the argument $\zeta = z-ct$. By integrating (3.2) once we can express g in terms of f:

$$g = -\frac{c(f-1)+(1-\phi_0)}{1-\phi_0 f}. \tag{3.4}$$

In arriving at this expression for the vertical velocity we have assumed that far from the disturbance the voidage ϕ tends to 1, which is the value in the uniform compacting background solution. Similarly, the velocity of the solid matrix tends to the uniform compacting velocity. In other words

$$\left.\begin{aligned} f &\to 1, \\ g &\to -1 \end{aligned}\right\} \quad \text{as } |\zeta| \to \infty. \tag{3.5}$$

By eliminating g from the problem, we are led to a single nonlinear ordinary differential equation:

$$\tfrac{1}{2}(1-\phi_0)(c+\phi_0)(1-\phi_0 f)^2 \frac{\mathrm{d}}{\mathrm{d}f}\left[\frac{p^2}{(1-\phi_0 f)^4}\right] - \frac{c(f-1)+(1-\phi_0)}{(1-\phi_0 f) f^n} + \frac{1-\phi_0 f}{1-\phi_0} = 0, \tag{3.6}$$

where $\qquad p = f'.$

We have exploited the fact that (3.2) and (3.3) are autonomous to eliminate ζ from the problem. In the phase plane (f, p), the solitary waves of permanent form correspond to the homoclinic trajectories that start at $(1, 0)$, form an arch in the first quadrant, cross the f-axis at $f = A$ and return to the point $(1, 0)$ in a symmetric fashion. To simplify the description of these trajectories, we define

$$C(f;\phi_0 n) = -\frac{\displaystyle\int_1^f \frac{(1-\phi_0)\,\mathrm{d}x}{x^n(1-\phi_0)^3} + \frac{1}{\phi_0(1-\phi_0)}\ln\frac{1-\phi_0 f}{1-\phi_0}}{\displaystyle\int_1^f \frac{(x-1)\,\mathrm{d}x}{x^n(1-\phi_0 f)^3}}. \tag{3.7}$$

The desired trajectories are then given by

$$\frac{p^2}{(1-\phi_0 f)^4} = \frac{2[c-C(f;\phi_0,n)]}{(1-\phi_0)(c+\phi_0)}\int_1^f \frac{(x-1)\,\mathrm{d}x}{x^n(1-\phi_0 x)^3}. \tag{3.8}$$

The relationship between the maximum amplitude A of a wave and its phase speed c is then

$$c = C(A;\phi_0,n). \tag{3.9}$$

In figure 3 we display this dependence of the phase speed on the amplitude for various choices of the far-field voidage ϕ_0. For a maximal-amplitude A, c increases with decreasing ϕ_0. The dependence of $C(A;\phi_0,n)$ on the exponent n of the power law between the permeability and voidage is shown in figure 4. It is comforting to see that $n = 3$ is typical of a range of values of the exponent.

To compute the shape of the solitary wave we introduce

$$P(f;A,\phi_0,n) = \frac{\surd(2)\,(1-\phi_0 f)^2}{(1-\phi_0)^{\frac{1}{2}}[C(A;\phi_0,n)+\phi_0]^{\frac{1}{2}}} \times\left\{(C(A;\phi_0,n)-C(f;\phi_0,n))\int_1^f \frac{(x-1)\,\mathrm{d}x}{x^n(1-\phi_0 x)^3}\right\}^{\frac{1}{2}}. \tag{3.10}$$

The shape is then given by the implicit function

$$\zeta = \int_f^A \frac{\mathrm{d}x}{P(x;A,\phi_0,n)}. \tag{3.11}$$

Since $P(x;A,\phi_0,n)$ vanishes for $x = A$, (3.11) is modified slightly for the purpose of numerical integration. Firstly, we Taylor expand P^2, viz

$$P^2(f;A,\phi_0,n) = 2\kappa(f-A)+\dots, \tag{3.12}$$

where
$$\kappa = \frac{1}{2}\frac{\partial P^2}{\partial f}\bigg|_{f=A} = \frac{(1-\phi_0 A)^2\left.\dfrac{\mathrm{d}C}{\mathrm{d}f}\right|_{f=A}}{(1-\phi_0)[\phi_0-C(A;\phi_0,n)]}\int_1^A \frac{(x-1)\,\mathrm{d}x}{x^n(1-\phi_0 x)}. \tag{3.13}$$

We then replace (3.11) with

$$\zeta = \int_f^{A-\epsilon} \frac{\mathrm{d}x}{P(x;A,\phi_0,n)} + \frac{\surd 2\,\epsilon^{\frac{1}{2}}}{|\kappa|}. \tag{3.14}$$

Incidentally, κ is the curvature of the shape of the solitary wave at the maximum height.

The procedure used to generate solitary waves such as those in figure 5 is as follows. Values of ϕ_0 and n are selected. Then $C(f;\phi_0,n)$ is tabulated for values of f between 1 and ϕ_0^{-1}; the values of $f_{\max}$ and $C_{\max}$ at which $\mathrm{d}C/\mathrm{d}f$ vanishes are recorded. Next a value of A in the range $(1, f_{\max})$ is selected. The speed $C(A;\phi_0,n)$ of the wave of this amplitude is read. Finally, $P(x;A,\phi_0,n)$ is computed for x in $(1, A)$ and then ζ

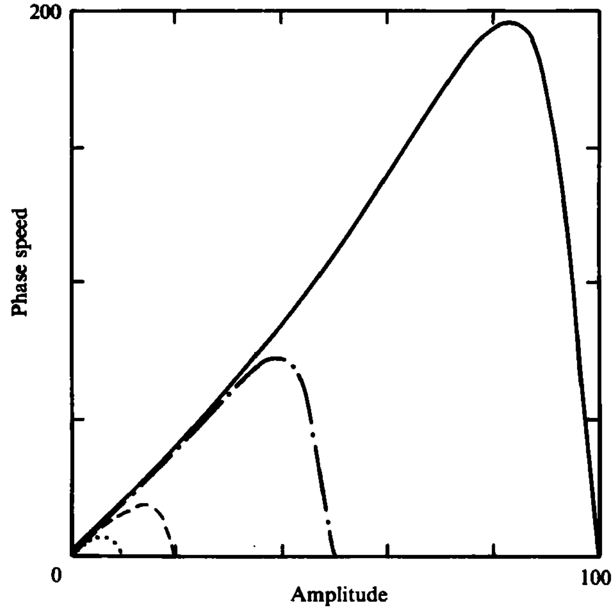

FIGURE 3. Phase speed of solitary waves as a function of their peak amplitude for different values of the background voidage ϕ_0: ——, $\phi_0 = 0.01$; —·—·—, 0.02; – – – –, 0.05; ...., 0.1. $n = 3$ in all cases.

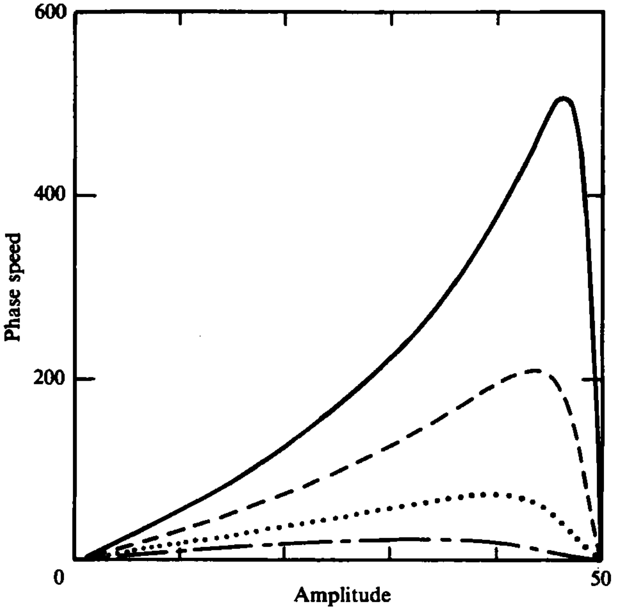

FIGURE 4. Phase speed of solitary waves as a function of their peak amplitude for different values of n, the exponent in the permeability–voidage relation: $n = 4$; – – – –, 3.5; ...., 3; —·—, 2.5. The background voidage ϕ_0 is 0.02 in all cases.

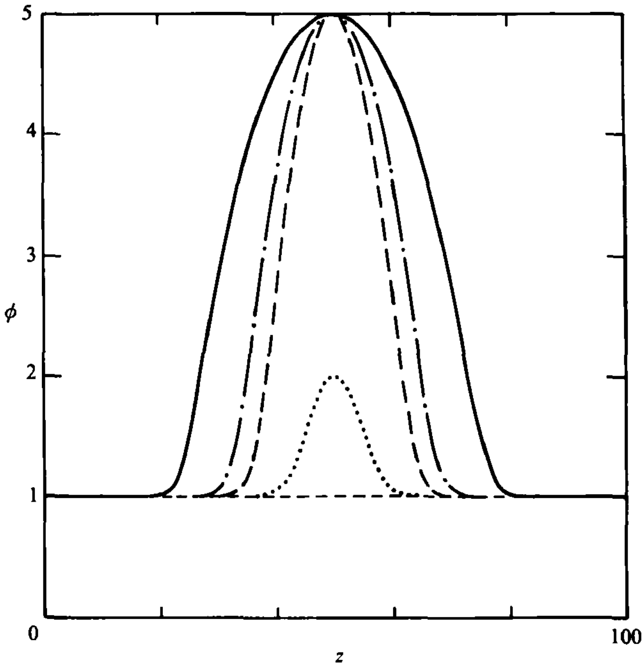

FIGURE 5. Shape of solitary waves for different choices of background voidage ϕ_0, exponent n and amplitude: ——, $\phi_0 = 0.05$, $n = 5$; —·—·—, $\phi_0 = 0.05$, $n = 3$; ------, $\phi_0 = 0.01$, $n = 3$; ...., $\phi_0 = 0.01$, $n = 3$.

is found by means of (3.14). The very last step consists in inverting this implicit relation between ζ and f. A measure of the accuracy of our numerical integration is obtained by using the solitary waves thus computed as initial conditions for integrating numerically the evolution equations. We find that these waves propagate without change of shape to within an accuracy of better than 0.1 % over distances of many pulse widths.

We have already remarked that κ is the negative curvature of the solitary wave at $\zeta = 0$. Since $\mathrm{d}C/\mathrm{d}f|_{f-A}$ tends to zero as A tends to $f_{\max}$, it is clear that the fastest wave is singular. Indeed, not only is

$$\left.\frac{\mathrm{d}f}{\mathrm{d}\zeta}\right|_{\zeta=0} = \left.\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2}\right|_{\zeta=0} = 0,$$

but *all* the derivatives of f vanish at $\zeta = 0$.

We conclude this section by considering the very useful approximation $\phi_0 \ll 1$, i.e. the small background melt. For this case and for $n = 3$, we can greatly simplify the analysis and arrive at analytical formulas for the phase speed and shape. Indeed, the governing equations (3.2) and (3.3) reduce to

$$\phi_t = W_z, \tag{3.15}$$

$$W_{zz} - \frac{W}{\phi^3} - 1 = 0, \tag{3.16}$$

or, in terms of ϕ alone,

$$\phi_t = [\phi^3(\phi_{zt} - 1)]_z. \tag{3.17}$$

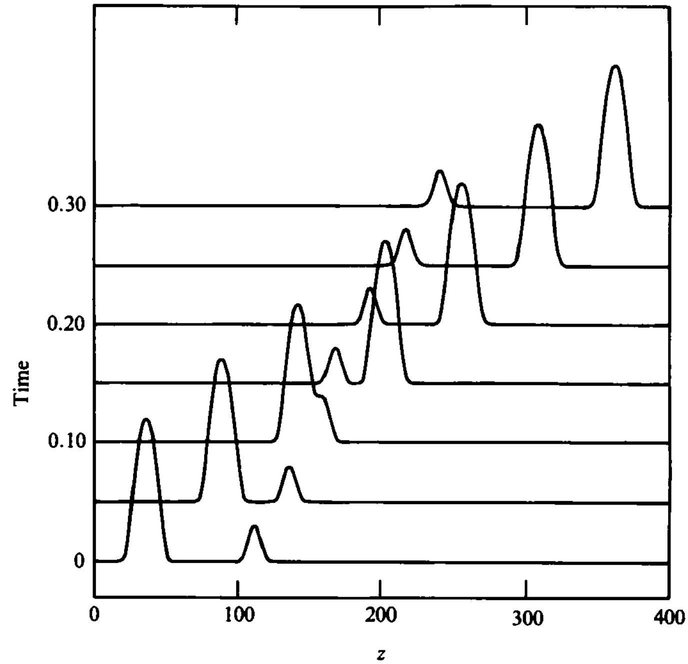

FIGURE 6. Nonlinear interaction of a solitary wave of amplitude 5 with one of amplitude 2. The background voidage $\phi_0 = 0.01$ and $n = 3$. The voidage distribution *vs* depth is plotted at a succession of times.

The counterpart of (3.6) for the trajectory in the phase plane is

$$\tfrac{1}{2}c\frac{\mathrm{d}p^2}{\mathrm{d}f}-\frac{c(f-1)+1}{f^3}+1=0, \tag{3.18}$$

and the analogue of (3.9) is
$$c=2A+1. \tag{3.19}$$

Finally the shape equation (3.11) reduces to

$$\zeta=\pm(A+\tfrac{1}{2})\left[-2(A-f)^{\frac{1}{2}}+\frac{1}{(A-1)^{\frac{1}{2}}}\ln\frac{(A-1)^{\frac{1}{2}}-(A-f)^{\frac{1}{2}}}{(A-1)^{\frac{1}{2}}+(A-f)^{\frac{1}{2}}}\right]. \tag{3.20}$$

The shape given by (3.20) differs little from the exact solution found when ϕ_0 terms are retained, yet when (3.20) is used as an initial condition for the numerical integration of the full equations a dispersive tail of order ϕ_0 is left behind. Thus it is important to use as exact an initial condition as possible, especially in the investigation of the interaction of two solitary waves.

4. Interaction of solitary waves

The designation of soliton has come to imply solitary waves that can pass through each other and emerge unscathed by the interaction (Drazin 1984, p. 3). Figure 6 shows what appear to be solitons; the only effect of the interaction is an advance (in the (z,t)-plane) of the wave with the larger amplitude and a slight retardation of the smaller wave. On closer inspection (see figure 7) we can see that a small dispersive tail has been left behind at the place where the waves interacted. This

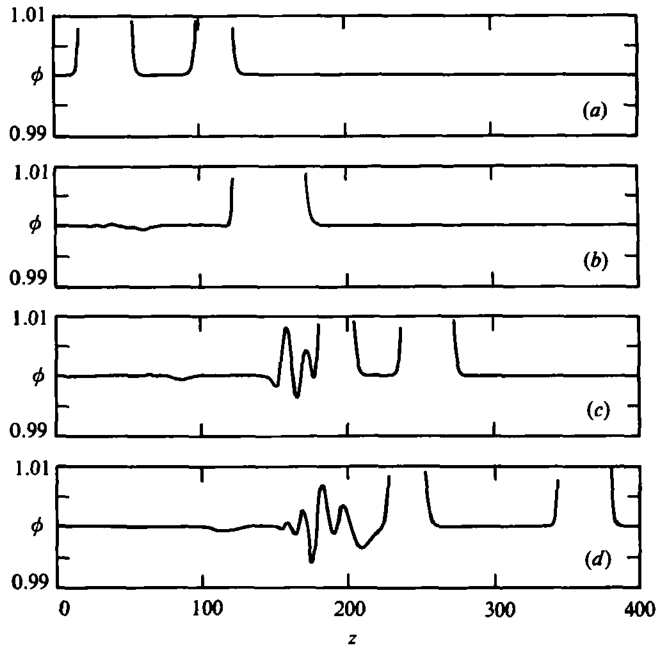

FIGURE 7. Magnification of the voidage around the background value, for the four times shown in figure 6, namely (*a*) $t = 0$, (*b*) 0.1, (*c*) 0.2 and (*d*) 0.3. The imperfection of the interaction of the solitary waves is clearly apparent.

result is very reminiscent of that found by Bona, Pritchard & Scott (1980) for the BBM equation (Benjamin, Bona & Mahony 1972), and suggests that the solitary waves under consideration are not solitons. This suspicion is strengthened further by the fact that (3.8) is not of the Painlevé form.

It is important that we be able to show that the imperfections of the interaction of the solitary waves are not due to an artifact of the numerical methods. To that effect, several runs have been made, as a result of which we can show that the dispersive waves seen in figure 7 are much larger than the error associated with the propagation of a single solitary wave. This can be seen in figure 8, which exhibits the larger solitary wave travelling alone; while there is a small error due to the discretization of the initial condition, it is about an order of magnitude smaller than the oscillations in figure 7. Secondly, figure 9 shows that the post-interactive wave is unchanged after the time step has been halved.

There is still no general test which can discriminate between solitons and ordinary solitary waves. However, one of the tenets of the theory of nonlinear wave propagation is that solitons possess an infinite number of conservation laws. In the next section, we shall outline a procedure for the systematic search for such conservation laws.

5. Conservation laws

In this section, we initiate a systematic search for conservation laws. This search is carried out on the simplest version of our problem, namely for the $\phi_0 \ll 1$ case, i.e. for the evolution equation (3.17), which we rewrite as follows:

$$\phi_{zzt} = \frac{\phi_t}{\phi^3} - 3\,\frac{\phi_z \phi_{zt}}{\phi} + 3\,\frac{\phi_z}{\phi}. \tag{5.1}$$

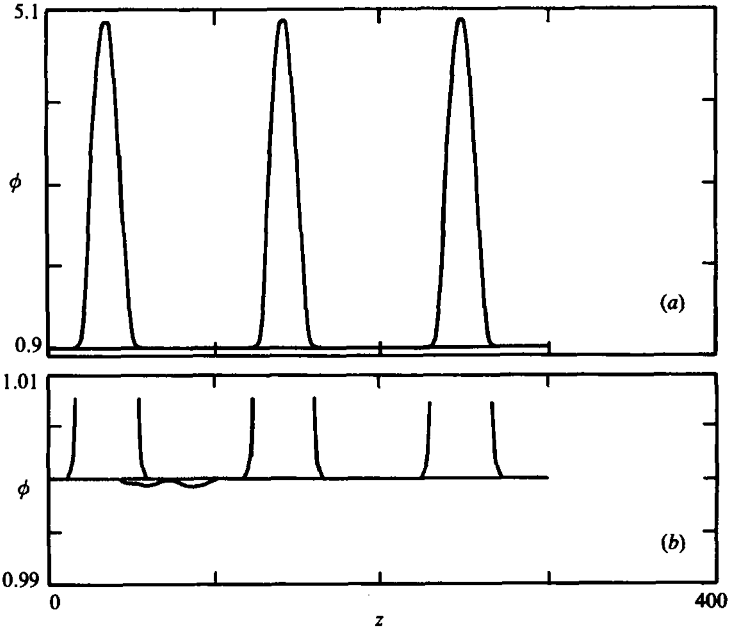

FIGURE 8. Propagation of a single solitary wave of amplitude 5 with $\phi_0 = 0.01$ and $n = 3$. Both the normal and magnified profiles at time $t = 0, 0.1, 0.2$ are identical with the corresponding profiles in figures 6 and 7. (*a*) Normal scale; (*b*) magnification of the voidage around the background value showing that the level of error is about an order of magnitude less than the oscillations seen in figure 7.

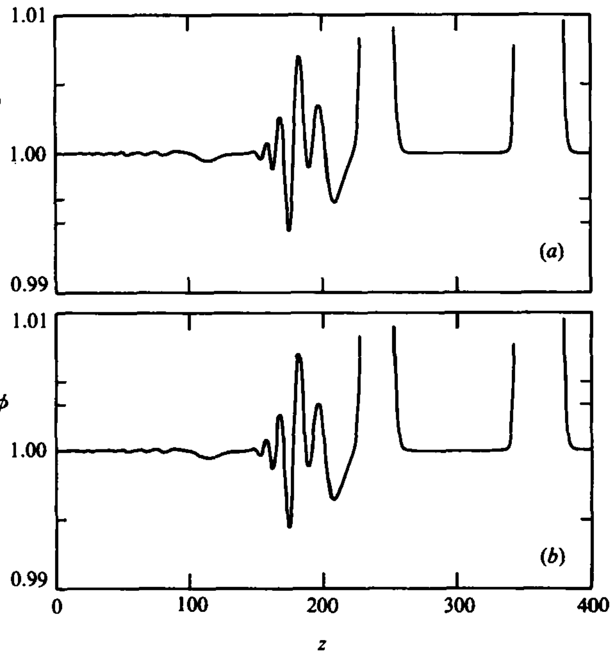

FIGURE 9. Magnification at $t = 0.3$ of the voidage ϕ around its background value ϕ_0 for the interaction shown in figure 6. (*a*) corresponds to figure 7 (*d*); (*b*) is the result when the time step used in the numerical calculations is increased by a factor of 2. The major oscillations left behind the interaction are essentially unchanged, suggesting that they have been well resolved by the numerical solution.

This equation is reminiscent of the BBM equation already alluded to, namely

$$\phi_{zzt} = \phi_t - \phi\phi_z. \tag{5.2}$$

For this equation, Olver (1979) has shown that there exist only three conservation laws. Because of the complicated form of our evolution equation, we shall not be able to arrive at a definitive conclusion about the actual number of conservation laws. However, the analysis which follows, as well as that in Appendix B, suggests that there are only two conservation laws for (5.1).

By conservation law for an evolution equation of the form (5.1), one means a pair of quantities

$$T_n(\phi, \phi_z, \ldots, \phi_{z\ldots z}, \ldots), \tag{5.3}$$

$$X_n(\phi, \phi_z, \ldots, \phi_{z\ldots z}, \ldots, \phi_t, \phi_{zt}), \tag{5.4}$$

such that

$$\frac{\partial T_n}{\partial t} = \frac{\partial X_n}{\partial z}. \tag{5.5}$$

Several remarks are in order at this stage. First, we draw attention to the fact that T_n depends on ϕ and possibly many z-derivatives of ϕ but *not* on t-derivatives of ϕ. This is because one ought to be able to evaluate T_n at $t = 0$; hence it should only be a function of the initial conditions. Indeed, by integrating (5.5) over z from $-\infty$ to $+\infty$, we get

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} T_n \, \mathrm{d}z = 0, \tag{5.6}$$

i.e. the integral of T_n is a conserved quantity. In contrast, X_n can depend on t-derivatives of ϕ. But, as (5.4) indicates, only the dependence on ϕ_t and ϕ_{zt} needs to be included. Indeed, ϕ_{zzt}, ϕ_{zzzt}, etc. can be expressed in terms of ϕ_t and ϕ_{zt} as well as z-derivatives of ϕ through (5.1). Finally, higher t-derivatives of ϕ cannot be balanced by any terms in the left-hand side of (5.5) and hence need not be included.

We can now state the results of our search. We have been able to obtain two conservation laws associated with functionals $T_0(\phi)$ and $T_1(\phi, \phi_z)$. In addition, we have been able to show that there are no conservation laws associated with functionals of the form $T_2(\phi, \phi_z, \phi_{zz})$ or $T_3(\phi, \phi_z, \phi_{zz}, \phi_{zzz})$. These results lead us to believe that there are only two conservation laws for (5.1).

We start by proving that X must be a linear function of ϕ_t and ϕ_{zt}. The proof follows that of Olver. Consider the generic pair

$$\left.\begin{aligned} T &= T(\phi, \phi_z, \ldots, \phi_{nz}), \\ X &= X(\phi, \phi_z, \ldots, \phi_{nz}, \phi_t, \phi_{zt}), \end{aligned}\right\} \tag{5.7}$$

where ϕ_{nz} stands for $\partial^n\phi/\partial z^n$. The conservation law (5.5) implies that

$$\phi_t \partial_0 T + \phi_{zt} \partial_1 T + \ldots + \phi_{nzt} \partial_n T = \phi_z \partial_0 X + \ldots + \phi_{(n+1)z} \partial_n X + \phi_{zt} \partial_p X + \phi_{zzt} \partial_q X, \tag{5.8}$$

where we have used

$$\partial_k = \partial/\partial\phi_{kz} \tag{5.9}$$

and

$$\partial_p = \partial/\partial\phi_t, \quad \partial_q = \partial/\partial\phi_{zt}. \tag{5.10}$$

As previously mentioned, we can write ϕ_{kzt} in terms of ϕ_t, ϕ_{zt}, and z-derivatives of ϕ by differentiating the evolution equation (5.1). Therefore

$$\phi_{kzt} = \alpha_k \phi_t + \beta_k \phi_{zt} + \gamma_k, \tag{5.11}$$

where
$$\left.\begin{aligned} \alpha_k &= \alpha_k(\phi, \phi_z, \ldots, \phi_{(k-2)z}), \\ \beta_k &= \beta_k(\phi, \phi_z, \ldots, \phi_{(k-1)z}), \\ \gamma_k &= \gamma_k(\phi, \phi_z, \ldots, \phi_{(k-1)z}). \end{aligned}\right\} \tag{5.12}$$

As a result, (5.8) reads

$$\begin{aligned} &\phi_t[\partial_0 T + \alpha_2 \partial_2 T + \ldots + \alpha_n \partial_n T] + \phi_{zt}[\partial_1 T + \beta_2 \partial_2 T + \ldots + \beta_n \partial_n T] \\ &\quad + [\gamma_2 \partial_2 T + \ldots + \gamma_n \partial_n T] \\ &\qquad = \phi_z \partial_0 X + \ldots + \phi_{(n+1)z} \partial_n X + \frac{\phi_t}{\phi^3} \partial_q X + \phi_{zt}\left(\partial_p X - 3\frac{\phi_z}{\phi} \partial_q X\right) + 3\frac{\phi_z}{\phi} \partial_q X. \end{aligned} \tag{5.13}$$

The dependence on ϕ_t and ϕ_{zt} is explicit on the left-hand side and hence is linear. Following Olver, we shall exploit this observation to narrow the functional form of X. This is done by differentiating (5.13): (i) twice with respect to ϕ_t; (ii) once with respect to ϕ_t and once with respect to ϕ_{zt}; and finally (iii) twice with respect to ϕ_{zt}. In the first instance, we get

$$0 = \partial_{pp}\left[\phi_z \partial_0 X + \ldots + \phi_{(n+1)z} \partial_n X + \frac{\phi_t}{\phi^3} \partial_q X + \phi_{zt}\left(\partial_p X - 3\frac{\phi_z}{\phi} \partial_q X\right) + 3\frac{\phi_z}{\phi} \partial_q X\right]; \tag{5.14}$$

in the second

$$0 = \partial_{pq}\left[\phi_z \partial_0 X + \ldots + \phi_{(n+1)z} \partial_n X + \frac{\phi_t}{\phi^3} \partial_q X + \phi_{zt}\left(\partial_p X - 3\frac{\phi_z}{\phi} \partial_q X\right) + 3\frac{\phi_z}{\phi} \partial_q X\right]; \tag{5.15}$$

and in the third

$$0 = \partial_{qq}\left[\phi_z \partial_0 X + \ldots + \phi_{(n+1)z} \partial_n X + \frac{\phi_t}{\phi^3} \partial_q X + \phi_{zt}\left(\partial_p X - 3\frac{\phi_z}{\phi} \partial_q X\right) + 3\frac{\phi_z}{\phi} \partial_q X\right]. \tag{5.16}$$

Note that in each of these equations the dependence on $\phi_{(n+1)z}$ is explicit. Therefore

$$\partial_{pp}(\partial_n X) = \partial_{pq}(\partial_n X) = \partial_{qq}(\partial_n X) = 0, \tag{5.17}$$

and, since the above holds for all values of $n \neq 0$, we conclude that

$$\begin{aligned} \partial_{pp} X &= P(\phi, \phi_t, \phi_{tz}), \\ \partial_{pq} X &= R(\phi, \phi_t, \phi_{tz}), \\ \partial_{qq} X &= Q(\phi, \phi_t, \phi_{tz}). \end{aligned}$$

We shall not make any use of these results other than to simplify (5.14)–(5.16), which become

$$0 = \phi_z \partial_{pp}\left[\partial_0 X - \frac{3}{\phi}(q-1)\partial_q X\right] + \partial_{pp}\left[q\,\partial_p X + \frac{p}{\phi^3}\partial_q X\right], \tag{5.18}$$

$$0 = \phi_z \partial_{pq}\left[\partial_0 X - \frac{3}{\phi}(q-1)\partial_q X\right] + \partial_{pq}\left[q\,\partial_p X + \frac{p}{\phi^3}\partial_q X\right], \tag{5.19}$$

$$0 = \phi_z \partial_{qq}\left[\partial_0 X - \frac{3}{\phi}(q-1)\partial_q X\right] + \partial_{qq}\left[q\,\partial_p X + \frac{p}{\phi^3}\partial_q X\right]. \tag{5.20}$$

We exploit next the explicit dependence on ϕ_z. Clearly,

$$\partial_0 X - \frac{3}{\phi}(q-1)\,\partial_q X = a_1 p + b_1 q + c_1, \tag{5.21}$$

$$q\,\partial_p X + \frac{p}{\phi^3}\,\partial_q X = a_2 p + b_2 q + c_2, \tag{5.22}$$

where a_i, b_i and c_i are arbitrary functions of $\phi, \phi_z, \ldots, \phi_{nz}$ but not $p = \phi_t$ or $q = \phi_{zt}$. If we were to integrate (5.21), we should find that

$$X = \int^{\phi} \left[p a_1(\xi, \phi_z, \ldots) + \left(\frac{q-1}{\xi^3}\,\phi^3 + 1\right) b_1(\xi, \phi_z, \ldots) + c_1(\xi, \phi_z, \ldots)\right] \mathrm{d}\xi + \mathscr{F}(\phi^3(q-1)), \tag{5.23}$$

whereas the integration of (5.22) implies that

$$X = a_2\,\phi^3 q + b_2\,p + c_2\,\phi^{\frac{3}{2}} \ln\left(p + \frac{q}{\phi^{\frac{3}{2}}}\right) + \mathscr{G}\left(q^2 - \frac{p^2}{\phi^3}\right), \tag{5.24}$$

where $\mathscr{F}$ and $\mathscr{G}$ are arbitrary functions. For these two expressions of X to be compatible, we must require that $\mathscr{F}$ be a linear function, that c_2 be identically zero and that $\mathscr{G}$ be a constant. In other words, X must have the following form:

$$X(\phi, \phi_z, \ldots, \phi_t, \phi_{zt}) = A(\phi, \phi_z, \ldots, \phi_{nz})\,\phi_t + B(\phi, \phi_z, \ldots, \phi_{nz})\,\phi_{zt} + C(\phi, \phi_z, \ldots, \phi_{nz}). \tag{5.25}$$

Thus the dependence of X on ϕ_t and ϕ_{zt} is at most linear.

In view of (5.25) and (5.13), the search for conservation laws has been reduced to the search for four functions T, A, B and C of $\phi, \phi_z, \ldots, \phi_{nz}$ such that

$$p\mathrm{L}_n T + q\mathrm{M}_n T + \mathrm{N}_n T = p\left(\mathrm{D}_n A + \frac{B}{\phi^3}\right) + q\left(\mathrm{D}_n B + A - 3\,\frac{\phi_z}{\phi}\,B\right) + \mathrm{D}_n C + 3\,\frac{\phi_z}{\phi}\,B, \tag{5.26}$$

where

$$\mathrm{L}_n = \partial_0 + \sum_{k=2}^{n} \alpha_k\,\partial_k,$$

$$\mathrm{M}_n = \partial_1 + \sum_{k=2}^{n} \beta_k\,\partial_k,$$

$$\mathrm{N}_n = \sum_{k=2}^{n} \gamma_k\,\partial_k,$$

and

$$\mathrm{D}_n = \sum_{k=0}^{n} \phi_{kz}\,\partial_k.$$

Because of the explicit dependence on p and q, (5.26) reduces to a system of three coupled first-order partial differential equations:

$$\mathrm{L}_n T = \mathrm{D}_n A + B\phi^{-3}, \tag{5.27}$$

$$\mathrm{M}_n T = \mathrm{D}_n B - 3\phi_z\,\phi^{-1} B + A, \tag{5.28}$$

$$\mathrm{N}_n T = \mathrm{D}_n C + 3\phi_z\,\phi^{-1} B. \tag{5.29}$$

We continue to chip away at the functional form of T, A, B and C by the same means as previously, namely by relying on the explicit dependence of certain variables. The obvious candidate at this stage is ϕ_{nz}. We have assumed that T, A, B and C depend on ϕ_{nz}. Actually, ϕ_{nz} can only enter in T. Indeed, if A, B and C were to depend on ϕ_{nz}, then a term linear in $\phi_{(n+1)z}$ would appear in the right-hand sides of (5.27)–(5.29) that could not be balanced by any other such term on the left-hand side. Therefore

$$\left.\begin{aligned} T &= T(\phi, \phi_z, \ldots, \phi_{nz}), \\ A &= A(\phi, \phi_z, \ldots, \phi_{(n-1)z}), \\ B &= B(\phi, \phi_z, \ldots, \phi_{(n-1)z}), \\ C &= C(\phi, \phi_z, \ldots, \phi_{(n-1)z}). \end{aligned}\right\} \tag{5.30}$$

Since $\alpha_k, \beta_k, \gamma_k$ do not depend on ϕ_{nz} for $k = 2, \ldots, n$ we can differentiate (5.27)–(5.29) once with respect to ϕ_{nz}:

$$\mathrm{L}_n F = \partial_{n-1} A, \tag{5.31}$$

$$\mathrm{M}_n F = \partial_{n-1} B, \tag{5.32}$$

$$\mathrm{N}_n F = \partial_{n-1} C, \tag{5.33}$$

where

$$F = \partial_n T. \tag{5.34}$$

Differentiating a second time, we get

$$\mathrm{L}_n G = 0, \tag{5.35}$$

$$\mathrm{M}_n G = 0, \tag{5.36}$$

$$\mathrm{N}_n G = 0, \tag{5.37}$$

where

$$G = \partial_n F. \tag{5.38}$$

We have arrived at three first-order partial differential equations for G, which is assumed to be a function of $n+1$ variables. The treatment of these equations differs according to whether $n+1$ is greater or smaller than 3. We refer the reader to Smirnov (1964, p. 357) for a lucid account of the theory of coupled first-order partial differential equations.

We consider here only the case $n = 2$, which subsumes the cases $n = 0$ and 1. The case $n = 3$ is considered in Appendix B. In other words, we search for conservation laws of the form $T = T(\phi, \phi_z, \phi_{zz})$. For this case, the system (5.35)–(5.37) has only the trivial solution (Smirnov 1964, p. 358)

$$G = G_0 = \text{constant}. \tag{5.39}$$

Consequently

$$T = \tfrac{1}{2} G_0 \phi_{zz}^2 + \Phi(\phi, \phi_z)\, \phi_{zz} + \Psi(\phi, \phi_z). \tag{5.40}$$

As a result (5.31)–(5.33) become

$$\partial_0 \Phi + \phi^{-3} G_0 = \partial_1 A, \tag{5.41}$$

$$\partial_1 \Phi - 3\phi_z \phi^{-1} G_0 = \partial_1 B, \tag{5.42}$$

$$3\phi_z \phi^{-1} G_0 = \partial_1 C, \tag{5.43}$$

from which we deduce that

$$\left.\begin{aligned} A &= \partial_0 \Omega - \frac{\phi_z^3}{2\phi^2} G_0, \\ B &= \partial_1 \Omega + \frac{1}{2\phi^2} G_0 + \hat{b}(\phi), \\ C &= \frac{3\phi_z^2}{2\phi} G_0 + \hat{c}(\phi), \\ \Phi &= \partial_1 \Omega + \frac{G_0}{2\phi^2} + \frac{3\phi_z^2 G_0}{2\phi}, \end{aligned}\right\} \tag{5.44}$$

where $\Omega = \Omega(\phi, \phi_z)$ is an arbitrary function at this stage. In order to deduce the form of Ψ we turn to (5.27)–(5.29) which read

$$\partial_0 \Psi + \phi^{-3}\Phi = \phi_z \partial_0 A + \phi^{-3}B, \tag{5.45}$$

$$\partial_1 \Psi - 3\phi_z \phi^{-1}\Phi = \phi_z \partial_0 B - 3\phi_z \phi^{-1}B + A, \tag{5.46}$$

$$3\phi_z \phi^{-1}\Phi = \phi_z \partial_0 C + 3\phi_z \phi^{-1}B. \tag{5.47}$$

After substituting the expressions for A, B, C and Φ in (5.45)–(5.47) we deduce that

$$G_0 = 0, \tag{5.48}$$

$$\hat{b} = -\tfrac{1}{2}a_0 \phi - b_0 \phi^3, \tag{5.49}$$

$$\hat{c} = \tfrac{3}{2}a_0 \phi + b_0 \phi^3, \tag{5.50}$$

$$\Psi = \phi_z \partial_0 \Omega + \frac{a_0}{2\phi} + \frac{a_0}{2}\phi_z^2 - b_0 \phi, \tag{5.51}$$

where a_0 and b_0 are arbitrary constants. In summary

$$T = \frac{\mathrm{d}}{\mathrm{d}z}[\Omega(\phi, \phi_z)] + \frac{a_0}{2}\left(\phi_z^2 + \frac{1}{\phi} - 1\right) - b_0(\phi - 1) \tag{5.52}$$

and

$$X = \frac{\mathrm{d}}{\mathrm{d}t}[\Omega(\phi\phi_z)] + \frac{a_0}{2}(-\phi\phi_{zt} + 3\phi) - b_0[\phi^3(f_{zt} - 1)]. \tag{5.53}$$

Obviously, we can discard the Ω-terms since they lead to trivial conservation laws. The remaining terms provide two conservation laws, namely

$$\left.\begin{aligned} T_0 &= \phi - 1, \\ X_0 &= \phi^3(\phi_{zt} - 1) \end{aligned}\right\} \tag{5.54}$$

and

$$\left.\begin{aligned} T_1 &= \phi_z^2 + \frac{1}{\phi} - 1, \\ X_1 &= -\phi\phi_{zt} + 3\phi. \end{aligned}\right\} \tag{5.55}$$

T_0 is related to the conservation of mass of the melt. The meaning of T_1 is not as clear; we should mention that the form of T_1 is closely related to the empirical power law for the permeability. For $K = \phi^n$,

$$T_1 = \phi_z^2 + \frac{2}{(n-1)(n-2)}\left\{\frac{1}{\phi^{n-2}} - 1\right\}. \tag{5.56}$$

6. Conclusions

We have investigated the consequences of the mathematical model proposed by McKenzie (1984) to explain the segregation and migration of melts. More particularly, we have concentrated on a study of the solitary waves that arise in this model.

The existence of compaction waves in this model, be they infinitesimal or of finite amplitude, is due to the prescribed relation between the voidage and the permeability. Equation (3.18), which contains the essential ingredients of this wave phenomenon, can be used to understand it. In addition to the conservation of mass, the crucial balance of forces in the vertical is one between the friction, pressure gradient and buoyancy. In their simplest form, these three forces are represented by W_{zz}, $-W/\phi^n$ and -1. In the steady state of compaction, a constant pressure gradient counteracts the buoyancy. Suppose that an increment in voidage occurs at a certain depth. This change to the basic state of uniform compaction results in an increase of the pressure gradient acting on the solid matrix. As a result, there is an increase in the downward flow of solid and a concomitant increase in the upward flow of melt. This upward flow in turn causes an increase in voidage at a depth slightly shallower than that of the initial disturbance.

As long as $n > 0$, this mechanism is responsible for the propagation of waves, both infinitesimal and of finite amplitude. However, finite-amplitude waves occur only for $n > 2$. Indeed, for $0 < n \leqslant 2$, the trajectories in phase space emanating from $(1, 0)$, which represents the steady compaction state, do not form closed curves. In this connection, we should also recall that the general conserved quantity T_2 is singular for $n = 2$.

We conclude these remarks by a speculation, namely our belief that, if such waves exist in geophysical situations, then they must be unstable to horizontal perturbations.

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Appendix A. Numerical methods

The general approach adopted is to use simple straightforward numerical techniques and run them at sufficiently high spatial and temporal resolution to obtain adequate accuracy for the purpose of this paper. The best measure of accuracy comes from using the calculated shapes for the solitary waves as initial conditions for the evolution equations and noting whether they propagate at the correct speed and without change in shape.

Equation (2.14) is discretized as

$$\phi_i^{n+1} = \phi_i^n + \frac{\Delta t}{\phi_0 \Delta z}\{(1-\phi_0 \phi_{i+\frac{1}{2}}^{n+\frac{1}{2}})\, W_{i+\frac{1}{2}}^{n+\frac{1}{2}} - (1-\phi_0 \phi_{i-\frac{1}{2}}^{n+\frac{1}{2}})\, W_{i-\frac{1}{2}}^{n+\frac{1}{2}}\}, \tag{A 1}$$

where the superscript indicates the time step and the subscript the vertical grid point. The ϕ and W grids are staggered such that we have the vertical velocity at the top and bottom of the volume elements for which ϕ represents the volume average of voidage. In computing the flux we need an estimate of ϕ at the boundaries of the volume element, namely $\phi_{i\pm\frac{1}{2}}$, which we obtain by interpolation. The estimates of ϕ and W at $n+\frac{1}{2}$ are obtained by iterating within a time step.

Equation (2.15) can be put in the form

$$W'' + r(z)\, W = s(z), \tag{A 2}$$

which is discretized (Hildebrand 1956. p. 241) as

$$(1+\tfrac{1}{12}\Delta z^2 r_{i+1})\, W_{i+1} - 2(1-\tfrac{5}{12}\Delta z\, r_i)\, W_i + (1+\tfrac{1}{12}\Delta z^2 r_{i-1})\, W_{i-1}$$
$$= \tfrac{1}{12}\Delta z^2 (s_{i+1} + 10 s_i + s_{i-1}) + O(\tfrac{1}{240}\Delta z^6\, W^{(4)}). \tag{A 3}$$

Given the boundary conditions that $W = -1$ at the top and bottom of the region of interest, the system of coupled equations is solved by inverting a tridiagonal matrix by standard methods.

The shape of a solitary wave, given its amplitude A, background voidage ϕ_0 and exponent n in the permeability–voidage relation, is found by first calculating

$C(f;\,\phi_0, n)$ from (3.7) for values of f in the range $(1, A)$. The integrals are approximated using Simpson's rule. Equation (3.10) yields an equation of the form

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} = P(f), \tag{A 4}$$

which we solve using a 4th-order Runge–Kutta method with the initial value given at a distance ϵ away from the peak of the solitary pulse. The value of f at ϵ is obtained by a Taylor expansion, using the second derivative given by (3.13).

Appendix B. Lack of conserved quantity of the form $T_3(\phi, \phi_z, \phi_{zz}, \phi_{zzz})$

The starting point of our analysis is the trio (5.35)–(5.37) of first-order partial differential equations for G. The explicit form of these equations is

$$\mathrm{L}_3 G \equiv \partial_0 G + \phi^{-3}\partial_2 G - 6\phi_z\phi^{-4}\partial_3 G = 0, \tag{B 1}$$

$$\mathrm{M}_3 G \equiv \partial_1 G - 3\phi_z\phi^{-1}\partial_2 G + [\phi^{-3} + 12\phi_z^2\phi^{-2} - 3\phi_{zz}\phi^{-1}]\,\partial_3 G = 0, \tag{B 2}$$

$$\mathrm{N}_3 G \equiv 3\phi_z\phi^{-1}\partial_2 G - [12\phi_z^2\phi^{-2} - 3\phi_{zz}\phi^{-1}]\,\partial_3 G = 0. \tag{B 3}$$

We form all three possible Poisson brackets, namely

$$[\mathrm{L}_3, \mathrm{M}_3] \equiv \mathrm{L}_3(\mathrm{M}_3 G) - \mathrm{M}_3(\mathrm{L}_3 G) = 3\phi_z\phi^{-2}\partial_2 G - [24\phi_z^2\phi^{-3} - 3\phi_{zz}\phi^{-2}]\,\partial_3 G = 0, \tag{B 4}$$

$$[\mathrm{L}_3, \mathrm{N}_3] \equiv \mathrm{L}_3(\mathrm{N}_3 G) - \mathrm{N}_3(\mathrm{L}_3 G) = -3\phi_z\phi^{-2}\partial_2 G - [3\phi^{-4} - 24\phi_z^2\phi^{-3} + 3\phi_{zz}\phi^{-2}]\,\partial_3 G = 0, \tag{B 5}$$

$$[\mathrm{M}_3, \mathrm{N}_3] \equiv \mathrm{M}_3(\mathrm{N}_3 G) - \mathrm{N}_3(\mathrm{M}_3 G) = 3\phi^{-1}\partial_2 G - 24\phi_z\phi^{-2}\partial_3 G = 0. \tag{B 6}$$

Adding (B 4) to (B 5) yields $$\partial_3 G = 0. \tag{B 7}$$

As a result $$\partial_0 G = \partial_1 G = \partial_2 G = 0 \tag{B 8}$$

and consequently $$G = G_0 = \text{constant}.$$

Therefore, if there exists a conserved quantity T_3, it must have the form

$$T_3 = \tfrac{1}{2}G_0\phi_{zzz}^2 + \Phi(\phi, \phi_z, \phi_{zz})\,\phi_{zzz} + \Psi(\phi, \phi_z, \phi_{zz}). \tag{B 9}$$

We next turn our attention to Φ, which satisfies the following equations:

$$\mathrm{L}_2\Phi + \alpha_3 G_0 = \partial_2 A, \tag{B 10}$$

$$\mathrm{M}_2\Phi + \beta_3 G_0 = \partial_2 B, \tag{B 11}$$

$$\mathrm{N}_2\Phi + \gamma_3 G_0 = \partial_2 C, \tag{B 12}$$

or, more explicitly,

$$\partial_0\Phi + \phi^{-3}\partial_2\Phi - 6\phi\phi^{-4}G_0 = \partial_2 A, \tag{B 13}$$

$$\partial_1\Phi - 3\phi_z\phi^{-1}\partial_2\Phi + [\phi^{-3} + 12\phi_z^2\phi^{-2} - 3\phi_{zz}\phi^{-1}]\,G_0 = \partial_2 B, \tag{B 14}$$

$$3\phi_z\phi^{-1}\partial_2\Phi - [12\phi_z^2\phi^{-2} - 3\phi_{zz}\phi^{-1}]\,G_0 = \partial_2 C. \tag{B 15}$$

Since these are three equations in four unknowns we solve for ϕ, A and B in terms of C. Omitting some of the algebra, we find that

$$\Phi = \frac{\phi C}{3\phi_z} + G_0\left\{-\frac{\phi_{zz}^2}{2\phi_z} + 4\,\frac{\phi_z\phi_{zz}}{\phi}\right\} + \tilde{K}, \tag{B 16}$$

$$\begin{aligned} A = \partial_0\left[\frac{\phi}{3\phi_z}\int^{\phi_{zz}} C(\phi,\phi_z,x)\,\mathrm{d}x + \phi_{zz}\tilde{K}\right] + \frac{C}{3\phi^2\phi_z} \\ + G_0\left\{-\frac{\phi_{zz}^2}{2\phi^3\phi_z} - 2\,\frac{\phi_z\phi_{zz}^2}{\phi^2} - 2\,\frac{\phi_z\phi_{zz}}{\phi^4}\right\} + \tilde{A}, \end{aligned} \tag{B 17}$$

$$B = \partial_1\left[\frac{\phi}{3\phi_z}\int^{\phi_{zz}} C(\phi,\phi_z,x)\,\mathrm{d}x + \phi_{zz}\tilde{K}\right] - C + G_0\left\{\frac{\phi_{zz}^3}{6\phi_z^2} + 2\,\frac{\phi_{zz}^2}{\phi} + \frac{\phi_{zz}}{\phi^3}\right\} + \tilde{B}. \tag{B 18}$$

In the above expressions, the tilde denotes fields which are solely functions of ϕ and ϕ_z. Note that with the above information

$$\begin{aligned} X = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\phi}{3\phi_z}\int^{\phi_{zz}} C(\phi,\phi_z,x)\,\mathrm{d}x\right] + \phi_{zz}\frac{\mathrm{d}\tilde{K}}{\mathrm{d}t} + G_0\left\{\phi_t\left(-\frac{\phi_{zz}^2}{2\phi^3\phi_z} - 2\,\frac{\phi_z\phi_{zz}^2}{\phi^2} - 2\,\frac{\phi_z\phi_{zz}}{\phi^4}\right)\right. \\ \left. + \phi_{zt}\left(\frac{\phi_{zz}^3}{6\phi_z^2} + 2\,\frac{\phi_{zz}^2}{\phi} + \frac{\phi_{zz}}{\phi^3}\right)\right\} + \tilde{A}\phi_t + \tilde{B}\phi_{zt}. \end{aligned} \tag{B 19}$$

Clearly, the term that is a total time derivative can be thrown out. Therefore, redefining the tilde-terms if need be, we can write

$$\left.\begin{aligned} A &= G_0\left\{-\frac{\phi_{zz}^2}{2\phi^3\phi_z} - 2\,\frac{\phi_z\phi_{zz}^2}{\phi^2} - 2\,\frac{\phi_z\phi_{zz}}{\phi^4}\right\} + \tilde{A}, \\ B &= G_0\left\{\frac{\phi_{zz}^3}{6\phi_z^2} + 2\,\frac{\phi_{zz}^2}{\phi} + \frac{\phi_{zz}}{\phi^3}\right\} + \tilde{B}, \\ C &= \tilde{C}, \\ \Phi &= G_0\left\{-\frac{\phi_{zz}^2}{2\phi_z} + 4\,\frac{\phi_z\phi_{zz}}{\phi}\right\}. \end{aligned}\right\} \tag{B 20}$$

The situation regarding the determination of Ψ is quite different. Indeed, we must now contend with three equations in one unknown, viz

$$\mathrm{L}_2\,\Psi + \alpha_3\,\Phi = \mathrm{D}_1\,A + \frac{B}{\phi^3}, \tag{B 21}$$

$$\mathrm{M}_2\,\Psi + \beta_3\,\Phi = \mathrm{D}_1\,B + A - 3\,\frac{\phi_z}{\phi}\,B, \tag{B 22}$$

$$\mathrm{N}_2\,\Psi + \gamma_3\,\Phi = \mathrm{D}_1\,C + 3\,\frac{\phi_z}{\phi}\,B. \tag{B 23}$$

After substituting the expressions for A, B, C and Φ previously obtained, we find that the equations (B 21)–(B 23) are incompatible unless

$$G_0 = 0.$$

Thus, there is no conserved quantity of the form $T_3(\phi,\phi_z,\phi_{zz},\phi_{zzz})$.

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